This question paper contains 5 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : 6716

Unique Paper Code : 32371501 HC

Name of the Paper : Stochastic Processes and Queuing Theory

Name of the Course : B.Sc. (Hons.) STATISTICS

Semester : V

Duration : 3 Hours Maximum Marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *five* questions in all.

Section I is *compulsory.*

Attempt *four* more questions, selecting *two* questions from each of the Sections II and III.

Use of simple calculator is allowed.

Section I

1. Attempt any *five* parts : 5×3 = 15

(*a*) In the classical ruin problem, what will be the effect of reducing the unit of stake from one to half, on the probability of ruin of the gambler ?

(*b*) If $\{N(t), t \geq 0\}$ is a Poisson process, then show the autocorrelation coefficient between $N(t)$ and $N(t+s)$ is $\sqrt{\frac{t}{t+s}}$.

(c) Define convolution of two sequences $\{a_k\}$ and $\{b_j\}$, where $a_k = P(X = k)$ and $b_j = P(Y = j)$ with X and Y being two non-negative, integral valued random variables. Find the probability generating function of the sum of two independent random variables.

(d) Let $X(t) = A_0 + A_1 t + A_2 t^2$, where A_i, $i = 0, 1, 2$ are uncorrelated random variables with mean 0 and variance 1. Is $\{X(t), t \in T\}$ covariance stationary ?

(e) Define the following :

(i) Closed set

(ii) Ergodic state.

(f) Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{1, 2\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Find the stationary distribution of given t.p.m.

(g) Let X_n, for n even, takes values +1 and −1 each with probability $\frac{1}{2}$, and for n odd, take values $\sqrt{a}, \frac{-1}{\sqrt{a}}$, with probabilities $\frac{1}{1+a}, \frac{a}{1+a}$ respectively (a is real number > −1 and ≠ 0, 1). Further let X_n's be independent. Is $\{X_n, n \geq 1\}$ strictly stationary ?

Section II

2\. (a) Let a fair coin be tossed indefinitely. Find the probability that two or more consecutive heads will not occur in n tosses. Also find the generating function of this event.

(b) Let $S_N = X_1 + X_2 + ... + X_N$, where N has Poisson distribution with mean a. If X_i's have *i.i.d.* Bernoulli distribution with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p = q$, then show that :

(i) S_N has Poisson distribution with mean ap.

(ii) The joint distribution of S_N and N has the probability mass function

$$Pr(N = n, S_N = y) = \frac{e^{-a} a^n p^y q^{n-y}}{y!(n-y)!} \text{ for}$$

$n = 0, 1, 2, ...; y = 0, 1, 2, ... n$

(iii) Cov $(N, S_N) = ap$. 7,8

3\. (a) Obtain probability generating function of the random variable X having the mass function.

$p_k = \frac{1}{(2)} q^{(|k|-1)}(1 - q)$; $k = ..., -3, -2, -1, 1, 2, 3, ...$ where $0 < q < 1$.

(b) Six boys Dev (D), Hemant (H), Jeet (J), Mohan (M), Sunil (S) and Tejas (T) play the game of catching a ball. If D has the ball, he is equally likely to throw it to H, M, S and T. If H gets the ball, he is equally likely to throw it to D, J, S, T. If S has the ball, he is equally likely to throw it to D, H, M and T. If either J or T gets the ball, they keep throwing it to each other. If M gets the ball, he runs away with it. Obtain the transition probability matrix and classify the states. 7,8

4\. (a) Define a persistent state and a transient state. Show that the state j is persistent iff

$$\sum_{n=0}^{\infty} p_{jj}^n = \infty$$

(*b*) Consider a Markov chain $\{X_n, n \geq 0\}$ with states 0 and 1 having transition probability matrix.

$$X_{n-1}\begin{matrix} \\ 0 \\ 1 \end{matrix}\overset{\displaystyle X_n}{\begin{matrix} 0 & 1 \\ \end{matrix}}\begin{pmatrix} 1-(1-c)p & (1-c)p \\ (1-c)(1-p) & (1-c)p+c \end{pmatrix}; \; 0 < p < 1; \; 0 \leq c \leq 1$$

with initial distribution $P\{X_0 = 1\} = p_1 = 1 - P\{X_0 = 0\}$. Show that correlation coefficient Corr $\{X_{n-k}, X_n\} = c^k$ for $0 < c < 1$. 7,8

Section III

(*a*) Divide the interval $[0, t]$ into a large number n of small intervals of length h and suppose that in each small interval, Bernoulli trials with probability of success λh and with probability of failure $(1 - \lambda h)$ are held. Show that the number of successes in an interval of length t is a Poisson process with mean λt. State the assumptions you make.

(*b*) Show that for the linear growth process, the second moment $M_2(t)$ satisfies the differential equation $M'_2(t) = 2(\lambda - \mu)M_2(t) + (\lambda + \mu)\, M(t)$. Further, show that variance is

$$\text{Var }(X(t)) = i\frac{\lambda+\mu}{\lambda-\mu}\; e^{(\lambda-\mu)t}\; (e^{(\lambda-\mu)t} - 1); \; \lambda \neq \mu$$

where i is the population size at $t = 0$. 7,8

(*a*) Define a Poisson process. State the postulates under which a count process will be a Poisson process.

(*i*) Show that random selection from a Poisson process yields a Poisson process.

(*ii*) If $N_1(t)$, $N_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then obtain the distribution of $N(t) = N_1(t) - N_2(t)$.

(*b*) Describe the classical ruin problem. Derive an expression for the expected duration of the game, which is finite. Obtain the limiting expression as $a \to \infty$. 7,8

7\. (*a*) In the case of (M/M/1); (N/FCFS) queuing model, derive the steady-state probability distribution and obtain the expressions for :

(*i*) Expected number of customers in the system

(*ii*) Expected number of customers in the queue

(*iii*) Expected waiting time in the system.

(*b*) A super market has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Calculate :

(*i*) the probability that the cashier is idle

(*ii*) the average number of customers in the queue

(*iii*) the average time a customer spends in the system. 7,8